

SELSIMILAR SOLUTIONS IN A SECTOR FOR A QUASILINEAR PARABOLIC EQUATION *

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Abstract. We study a two-point free boundary problem in a sector for a quasilinear parabolic equation. The inhomogeneous boundary conditions are assumed to be spatially and temporally “similar” in a special way. We prove the existence and uniqueness of an expanding solution which is selfsimilar at discrete times, and then prove the existence of a shrinking solution which is selfsimilar at discrete times.

Key words. Selfsimilar solutions, parabolic equations, inhomogeneous boundary conditions

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1 Introduction

Consider the problem

$$(1.1) \quad \begin{cases} u_t = a(u_x)u_{xx}, & -\xi_1(t) < x < \xi_2(t), \quad t > 0, \\ u_x(x, t) = -k_1(t, u(x, t)), \quad u(x, t) = -x \tan \beta & \text{for } x = -\xi_1(t), \quad t > 0, \\ u_x(x, t) = k_2(t, u(x, t)), \quad u(x, t) = x \tan \beta & \text{for } x = \xi_2(t), \quad t > 0, \end{cases}$$

where $a \in C^2(\mathbb{R})$, $a(\cdot) > 0$, $\beta \in (0, \frac{\pi}{2})$ and $k_1, k_2 \in C^2([0, \infty) \times [0, \infty), \mathbb{R})$. In this problem, the triple u, ξ_1, ξ_2 are unknown (positive) functions to be determined.

Our motivation for studying this problem arises in flame propagation, motion by mean curvature and other applications (cf. [2, 3, 4, 7, 8, 9, 10, 11] etc.). [2, 3, 9, 10, 11] etc. considered the linear boundary value problem:

$$(1.2) \quad \begin{cases} u_t = a(u_x)u_{xx}, & -\zeta_1(t) < x < \zeta_2(t), \quad t > 0, \\ u_x(x, t) = -\gamma_1, \quad u(x, t) = -x \tan \beta & \text{for } x = -\zeta_1(t), \quad t > 0, \\ u_x(x, t) = \gamma_2, \quad u(x, t) = x \tan \beta & \text{for } x = \zeta_2(t), \quad t > 0, \end{cases}$$

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where γ_1, γ_2 are constants. They gave details on the a priori estimates and on the existence of solutions of (1.2) with some initial data. Moreover, [2, 3, 11] proved that when $\gamma_1 + \gamma_2 > 0$, any time-global solution u is expanding (that is, it moves upward to infinity) and it converges to a *selfsimilar solution*: $\sqrt{2t} \varphi(x/\sqrt{2t})$. On the other hand, [9, 10] proved that if $\gamma_1 + \gamma_2 < 0$ then any solution shrinks to 0 as $t \rightarrow T$ for some $T > 0$. Furthermore, if a is analytic then the rescaled solution $u/\sqrt{2(T-t)}$ converges to a *backward/shrinking selfsimilar solution* with the form: $\psi\left(x/\sqrt{2(T-t)}\right)$ as $t \rightarrow T$.

In this paper we will extend their results to spatially and temporally inhomogeneous boundary conditions: $k_1, k_2 \not\equiv \text{const.}$. Our boundary conditions mean that the contact angles between the curve (the graph of the solution) and the boundaries of the sector domain depend on the spatial and temporal environment. Clearly, selfsimilar functions like $\sqrt{2t} \varphi(x/\sqrt{2t})$ or $\sqrt{2(T-t)} \psi\left(x/\sqrt{2(T-t)}\right)$ is no longer a solution of (1.1). We have to adopt new concepts for the analogue of selfsimilar solutions. Our results in this paper show that problem (1.1) has a unique expanding solution which is selfsimilar at discrete times if k_1, k_2 have special “similarity” (see (1.5)) and if $\min k_1 + \min k_2 > 0$. On the other hand, problem (1.1) has a shrinking solution which is selfsimilar at discrete times if k_1, k_2 have special “similarity” (see (1.10)) and if $\max k_1 + \max k_2 < 0$.

Definition 1.1 *Let $(u, \xi_1, \xi_2) = (U, \Xi_1, \Xi_2)$ be a solution of (1.1) for $t > 0$. It is called an expanding selfsimilar solution if*

$$(1.3) \quad bU(x, t) \equiv U(bx, b^2t) \quad \text{for } -\Xi_1(t) \leq x \leq \Xi_2(t), \quad t > 0,$$

for some $b > 1$, and if

$$(1.4) \quad b\Xi_i(t) = \Xi_i(b^2t) \quad \text{for } t > 0 \quad (i = 1, 2).$$

From (1.3) we see that, for any $t_0 > 0$,

$$\dots = bU(b^{-1}x, b^{-2}t_0) = U(x, t_0) = b^{-1}U(bx, b^2t_0) = \dots.$$

This means that $U(x, t)$ is similar to $U(x, t_0)$ only at *discrete* times: $t = b^{2m}t_0$ ($m \in \mathbb{Z}$). In this sense we may also say that (U, Ξ_1, Ξ_2) (or, just U) is a *discrete selfsimilar solution* and $\sqrt{2t} \varphi(x/\sqrt{2t})$ is a *classical selfsimilar solution*.

It is easily seen that a necessary condition for the existence of a discrete expanding selfsimilar solution is that k_1 and k_2 are “similar” in a special way:

$$(1.5) \quad k_i(t, u) = k_i(b^2t, bu) \quad \text{for } t, u \geq 0.$$

For simplicity, we also impose another technical conditions on k_i :

$$(1.6) \quad |k_i(t, u)| < \tan \beta \quad \text{for } t, u \geq 0 \quad (i = 1, 2).$$

Theorem 1.1 *Assume k_1, k_2 satisfy conditions (1.5), (1.6). Assume also*

$$(1.7) \quad \min k_1 + \min k_2 > 0.$$

Then problem (1.1) has an expanding selfsimilar solution (U, Ξ_1, Ξ_2) .

Moreover, the selfsimilar solution is unique and $U_t > 0$ if $k_i \equiv k_i(u)$ ($i = 1, 2$).

Next we consider selfsimilar solutions which shrink to 0 in finite time.

Definition 1.2 *Given $T > 0$. Let $(u, \xi_1, \xi_2) = (\tilde{U}, \tilde{\Xi}_1, \tilde{\Xi}_2)$ be a solution of (1.1) for $t \in [0, T)$. If $\|\tilde{U}(\cdot, t)\|_{L^\infty} \rightarrow 0, \tilde{\Xi}_1(t) \rightarrow 0, \tilde{\Xi}_2(t) \rightarrow 0$ as $t \rightarrow T - 0$,*

$$(1.8) \quad \tilde{U}(x, t) \equiv b \tilde{U}(b^{-1}x, b^{-2}t + (1 - b^{-2})T) \quad \text{for } -\tilde{\Xi}_1(t) \leq x \leq \tilde{\Xi}_2(t), \quad 0 \leq t < T,$$

for some $b > 1$, and if

$$(1.9) \quad \tilde{\Xi}_i(t) = b \tilde{\Xi}_i(b^{-2}t + (1 - b^{-2})T) \quad \text{for } 0 \leq t < T \quad (i = 1, 2).$$

Then $(\tilde{U}, \tilde{\Xi}_1, \tilde{\Xi}_2)$ (or, just \tilde{U}) is called a shrinking (or, backward) selfsimilar solution of (1.1) on time interval $[0, T)$.

A necessary condition for the existence of such a solution is that

$$(1.10) \quad k_i(t, u) = k_i(b^{-2}t + (1 - b^{-2})T, b^{-1}u) \quad \text{for } 0 \leq t < T, \quad u \geq 0.$$

Replacing t by $T - t'$ we see that (1.8), (1.9) and (1.10) are equivalent to

$$(1.11) \quad \tilde{U}(x, T - t') \equiv b \tilde{U}(b^{-1}x, T - b^{-2}t') \quad \text{for } -\tilde{\Xi}_1(T - t') \leq x \leq \tilde{\Xi}_2(T - t'), \quad 0 < t' \leq T,$$

$$(1.12) \quad \tilde{\Xi}_i(T - t') = b \tilde{\Xi}_i(T - b^{-2}t') \quad \text{for } 0 < t' \leq T \quad (i = 1, 2),$$

and

$$(1.13) \quad k_i(T - t', u) = k_i(T - b^{-2}t', b^{-1}u) \quad \text{for } 0 < t' \leq T, \quad u \geq 0 \quad (i = 1, 2),$$

respectively.

Since $\tilde{U}(x, t)$ is similar to $\tilde{U}(x, t_0)$ only at discrete times: $t = b^{-2m}t_0 + (1 - b^{-2m})T$ ($m \in \mathbb{Z}$ and $2m \geq \log(\frac{T-t_0}{T})/\log b$), we may also say that $(\tilde{U}, \tilde{\Xi}_1, \tilde{\Xi}_2)$ is a *discrete shrinking selfsimilar solution* on $[0, T)$.

Theorem 1.2 *Given $T > 0$, assume k_1, k_2 satisfy condition (1.10) or (1.13). Assume also (1.6) and*

$$(1.14) \quad \max k_1 + \max k_2 < 0.$$

Then problem (1.1) has a shrinking selfsimilar solution on $[0, T)$.

Generally, there are no uniqueness results for the shrinking selfsimilar solutions (even for linear boundary value problem (1.2), (cf. [9, 10])).

Definition 1.2 and Theorem 1.2 discuss shrinking solutions on finite interval $[0, T)$. If we take a time shift, these solutions can be regarded as solutions on $[-T, 0)$. More precisely, let $(\tilde{U}, \tilde{\Xi}_1, \tilde{\Xi}_2)$ be a shrinking selfsimilar solution of (1.1) on $[0, T)$. Then

$$\hat{U}(x, t; T) := \tilde{U}(x, T + t) \quad \text{for } -\hat{\Xi}_1(t; T) \leq x \leq \hat{\Xi}_2(t; T), \quad t \in [-T, 0),$$

and

$$\hat{\Xi}_i(t; T) := \tilde{\Xi}_i(x, T + t) \quad \text{for } t \in [-T, 0) \quad (i = 1, 2)$$

satisfy

$$(1.15) \quad \hat{U}(x, t) \equiv b\hat{U}(b^{-1}x, b^{-2}t) \quad \text{for } -\hat{\Xi}_1(t) \leq x \leq \hat{\Xi}_2(t), \quad -T \leq t < 0,$$

and

$$(1.16) \quad \hat{\Xi}_i(t) = b\hat{\Xi}_i(b^{-2}t) \quad \text{for } -T \leq t < 0 \quad (i = 1, 2).$$

So $(\hat{U}, \hat{\Xi}_1, \hat{\Xi}_2)$ (which is defined on $[-T, 0)$ and shrinks to 0 as $t \rightarrow 0 - 0$) is a selfsimilar solution of

$$(1.17) \quad \begin{cases} u_t = a(u_x)u_{xx}, & -\xi_1(t) < x < \xi_2(t), \quad t < 0, \\ u_x(x, t) = -k_1(T + t, u(x, t)), & u(x, t) = -x \tan \beta \quad \text{for } x = -\xi_1(t), \quad t < 0, \\ u_x(x, t) = k_2(T + t, u(x, t)), & u(x, t) = x \tan \beta \quad \text{for } x = \xi_2(t), \quad t < 0. \end{cases}$$

We now consider shrinking selfsimilar solutions defined in $(-\infty, 0)$.

Definition 1.3 *Assume $k_i \equiv k_i(u)$ ($i = 1, 2$). Let $(\hat{U}, \hat{\Xi}_1, \hat{\Xi}_2)$ be a solution of (1.17) for $t \in (-\infty, 0)$. If it satisfies (1.15) and (1.16) for some $b > 1$ and $t \in (-\infty, 0)$, then it is called a shrinking (or, backward) selfsimilar solution of (1.17) in $(-\infty, 0)$.*

Theorem 1.3 *Assume $k_1 \equiv k_1(u)$ and $k_2 \equiv k_2(u)$ satisfy (1.6), (1.14) and $k_i(u) = k_i(bu)$ for some $b > 1$ and any $u \geq 0$. Then problem (1.17) has a shrinking selfsimilar solution in $(-\infty, 0)$.*

Our theorems extend the results about classical selfsimilar solutions in [2, 3, 9, 10, 11] to nonlinear boundary value problems. Our approach is essentially different from theirs though we will use their classical selfsimilar solutions as lower and upper solutions to give the growth bound for the solution of (1.1). We will convert the original problem by changing variables to some new unknowns in a flat cylinder, and then consider the ω -limits of the new unknowns. Using a result in [1] we show that for the problem of the new unknowns there are periodic solutions, which correspond to selfsimilar solutions of (1.1).

In Section 2 we give some preliminaries, including the selection of the initial data, comparison principles and the local existence result. In Section 3 we consider the expanding case and prove Theorem 1.1. In Section 4 we consider the shrinking case and prove Theorems 1.2 and 1.3.

2 Preliminaries

We use notation $S := \{(x, y) \mid y > |x| \tan \beta, x \in \mathbb{R}\}$, and use $\partial_1 S$ and $\partial_2 S$ to denote the left and right boundaries of S , respectively. Hereafter, for two functions $u_1(x), u_2(x)$ with $(x, u_1(x)) \in \overline{S}$, when we write

$$u_1(x) \leq u_2(x)$$

we indeed compare them on their common existence interval; when we write

$$u_1(x) \preceq u_2(x)$$

we mean that $u_1(x) \leq u_2(x)$ and the “equality” holds at some x .

For any $t_0 > 0$, let $u(x, t)$ be a classical solution of (1.1) on time interval $[0, t_0]$ with some initial datum. Denote

$$Q_{t_0} := \{(x, t) \mid -\xi_1(t) < x < \xi_2(t) \text{ and } 0 < t \leq t_0\}.$$

2.1 Initial data

Consider the problem (1.1) with initial datum

$$(2.1) \quad u(x, 0) = u_0(x), \quad -\xi_{01} \leq x \leq \xi_{02},$$

where $\xi_{0i} = \xi_i(0)$ ($i = 1, 2$), $u_0 \in C^{2+\mu}[-\xi_{01}, \xi_{02}]$ ($\mu \in (0, 1)$) with $u_0(\cdot) > 0$ and it satisfies the compatibility conditions:

$$(2.2) \quad (u_0)_x(-\xi_{01}) = -k_1(0, u_0(-\xi_{01})), \quad (u_0)_x(\xi_{02}) = k_2(0, u_0(\xi_{02})).$$

Since our main purpose in this paper is to construct selfsimilar solutions, we will not focus on general solutions of (1.1)-(2.1) with general initial data as it was done in [2, 9, 11]. We choose u_0 from $C^{2+\mu}$ space and only consider classical solution u of (1.1)-(2.1), which means that $u \in C^{2+\mu, 1+\mu/2}(\overline{Q}_{t_0})$ for some $t_0 > 0$. Furthermore, we always choose u_0 such that

$$(2.3) \quad -\tan \beta + \sigma \leq (u_0)_x \leq \tan \beta - \sigma \quad \text{for } -\xi_{01} \leq x \leq \xi_{02},$$

for some $\sigma \in (0, \tan \beta)$. This does not contradicts the compatibility conditions since condition (1.6) implies that for some $\sigma \in (0, \tan \beta)$, there holds

$$(2.4) \quad -\tan \beta + \sigma \leq k_i^0 := \min k_i < K_i^0 := \max k_i \leq \tan \beta - \sigma \quad (i = 1, 2).$$

2.2 Comparison principle

Definition 2.1 Let $u_1(x, t)$, $u_2(x, t)$ ($0 \leq t \leq t_0$) be two functions satisfying $(x, u_i(x, t)) \in \overline{S}$ ($i = 1, 2$). u_1 is called a lower solution of (1.1) if

$$(2.5) \quad \begin{cases} u_{1t} \leq a(u_{1x})u_{1xx} & \text{for } 0 \leq t \leq t_0 \text{ and } x \text{ with } (x, u_1(x, t)) \in S, \\ u_{1x}(-x, t) \geq -k_1(t, u_1(x, t)) & \text{for } 0 \leq t \leq t_0 \text{ and } x \text{ with } (x, u_1(x, t)) \in \partial_1 S, \\ u_{1x}(x, t) \leq k_2(t, u_1(x, t)) & \text{for } 0 \leq t \leq t_0 \text{ and } x \text{ with } (x, u_1(x, t)) \in \partial_2 S. \end{cases}$$

u_2 is called an upper solution of (1.1) if the opposite inequalities hold.

The following comparison principle follows from the maximum principle easily.

Lemma 2.1 Let $t_0 > 0$. Assume that $u_1(x, t)$ and $u_2(x, t)$ are lower solution and upper solution of (1.1) on $[0, t_0]$, respectively. If $u_1(x, 0) \leq u_2(x, 0)$, then $u_1(x, t) \leq u_2(x, t)$ for $0 \leq t \leq t_0$ and x with $(x, u_i(x, t)) \in \overline{S}$. If $u_1(x, 0) \leq u_2(x, 0)$ and $u_1(x, 0) \not\equiv u_2(x, 0)$, then $u_1(x, t) < u_2(x, t)$ for $0 < t \leq t_0$ and x with $(x, u_i(x, t)) \in \overline{S}$.

In the following sections we will use classical selfsimilar solutions of (1.2) as upper and lower solutions of (1.1) to give the growth bound for the solution u of (1.1)-(2.1).

2.3 Gradient bound of u

Lemma 2.2 *Let $u(x, t) \in C^{2+\mu, 1+\mu/2}(\overline{Q}_{t_0})$ be a solution of (1.1)-(2.1) on $[0, t_0]$. Then*

$$(2.6) \quad |u_x(x, t)| \leq \tan \beta - \sigma \quad \text{for } (x, t) \in \overline{Q}_{t_0}.$$

Proof. By (1.6), or (2.4) we have

$$u_x(\xi_2(t), t) = k_2(t, u(\xi_2(t), t)) \leq \tan \beta - \sigma,$$

and

$$u_x(-\xi_1(t), t) = -k_1(t, u(-\xi_1(t), t)) \leq -k_1^0 \leq \tan \beta - \sigma.$$

Combining these inequalities with (2.3) we obtain $u_x \leq \tan \beta - \sigma$ by maximum principle. $u_x \geq -\tan \beta + \sigma$ is proved similarly. \square

Corollary 2.1 *Let $u(x, t) \in C^{2+\mu, 1+\mu/2}(\overline{Q}_{t_0})$ be a solution of (1.1)-(2.1) on $[0, t_0]$. Then, for any $\theta \in [-\theta_0, \theta_0]$ with $\theta_0 := \frac{\pi}{2} - \beta$,*

$$(2.7) \quad \sigma \cos \beta \leq (1 \pm u_x(x, t) \tan \theta) \cos \theta \leq 2 - \sigma \cot \beta \quad \text{for } (x, t) \in \overline{Q}_{t_0}.$$

2.4 Change of variables

To study the local and global existence of solutions of the initial boundary value problem (1.1)-(2.1), it is convenient to introduce new coordinates that convert the sector domain S into a flat cylinder. More precisely, we will make a change of variables $(x, y, t) \mapsto (\theta, \rho, s)$, which gives a diffeomorphism $(\overline{S} \setminus \{0\}) \times [0, t_\infty) \rightarrow \overline{D} \times [s_0, s_\infty)$, where

$$D := \{(\theta, \rho) \in \mathbb{R}^2 \mid -\theta_0 < \theta < \theta_0, -\infty < \rho < \infty\}$$

with $\theta_0 := \frac{\pi}{2} - \beta$. The functions $\theta = \theta(x, y, t)$, $\rho = \rho(x, y, t)$ and $s = s(t)$ are to be specified below. With these new coordinates, the function $y = u(x, t)$ is expressed as $\rho = \omega(\theta, s)$, where the new unknown $\omega(\theta, s)$ is determined by the relation

$$(2.8) \quad \rho(x, u(x, t), t) = \omega(\theta(x, u(x, t), t), s(t)).$$

The function $\omega(\theta, s)$ is well-defined provided that the map $t \mapsto s(t)$ is strictly monotone for $t \in [0, t_\infty)$ and $x \mapsto \theta(x, u(x, t), t)$ is strictly monotone for each fixed

$t \in [0, t_\infty)$. We will see later that these monotonicity conditions always hold for the class of solutions which we consider. Indeed we will prove

$$(2.9) \quad \frac{\partial}{\partial t} s(t) > 0, \quad \frac{\partial}{\partial x} \theta(x, u(x, t), t) = \theta_x + \theta_y u_x > 0.$$

Once $\omega(\theta, s)$ is defined, then substituting it into the relation $y = u(x, t)$ yields

$$(2.10) \quad Y(\theta, \omega(\theta, s), s) = u(X(\theta, \omega(\theta, s), s), T(s)),$$

where the map $(\theta, \rho, s) \mapsto (X(\theta, \rho, s), Y(\theta, \rho, s), T(s)) : \overline{D} \times [s_0, s_\infty) \rightarrow (\overline{S} \setminus \{0\}) \times [0, t_\infty)$ is the inverse map of $(x, y, t) \mapsto (\theta(x, y, t), \rho(x, y, t), s(t))$. The expression (2.10) gives a formula for recovering the original solution $u(x, t)$ from $\omega(\theta, s)$. In order for u to be smoothly dependent on ω , we need the map $\theta \mapsto X(\theta, \omega(\theta, s), s)$ to be one-to-one for each fixed s and that $s \mapsto T(s)$ is strictly monotone for $s \in [s_0, s_\infty)$. Indeed we will prove

$$(2.11) \quad \frac{\partial}{\partial s} T(s) > 0, \quad \frac{\partial}{\partial \theta} X(\theta, \omega(\theta, s), s) = X_\theta + X_\rho \omega_\theta > 0.$$

2.5 Local existence

To get the local existence we first make the following change of variables.

$$(2.12) \quad \begin{cases} \theta = \arctan \frac{x}{y}, & (x, y) \in \overline{S} \setminus \{0\}, \\ \rho = \frac{1}{2} \log(x^2 + y^2), & (x, y) \in \overline{S} \setminus \{0\}, \\ s = t, & t \geq 0. \end{cases}$$

The inverse map is

$$(2.13) \quad \begin{cases} x = e^\rho \sin \theta, & (\theta, \rho) \in \overline{D}, \\ y = e^\rho \cos \theta, & (\theta, \rho) \in \overline{D}, \\ t = s, & s \in [0, \infty). \end{cases}$$

Clearly, $\theta = \theta_0$ and $\theta = -\theta_0$ correspond to $\partial_2 S$ and $\partial_1 S$, respectively.

Let $u(x, t) > 0$ be a classical solution of (1.1)-(2.1) for $t \geq 0$, then

$$(2.14) \quad \rho(x, u(x, t), t) = \omega(\theta(x, u(x, t), t), s) \Leftrightarrow e^{\omega(\theta, s)} \cos \theta = u(e^{\omega(\theta, s)} \sin \theta, t).$$

defines a new unknown $\rho = \omega(\theta, s)$ for $s \geq 0$. This function is well-defined since

$$\frac{\partial}{\partial x} \theta(x, u(x, t), t) = \frac{\partial}{\partial x} \left(\arctan \frac{x}{u(x, t)} \right) = \frac{1 - u_x \tan \theta}{u \cdot (1 + \tan^2 \theta)} > 0$$

by (2.7).

Differentiating the expression $e^{\omega(\theta,s)} \cos \theta = u(e^{\omega(\theta,s)} \sin \theta, t)$ twice by θ and once by t we obtain

$$u_x = \frac{\omega_\theta \cos \theta - \sin \theta}{\cos \theta + \omega_\theta \sin \theta}, \quad u_{xx} = \frac{\omega_{\theta\theta} - \omega_\theta^2 - 1}{e^\omega (\cos \theta + \omega_\theta \sin \theta)^3}, \quad u_t = \frac{e^\omega \omega_t}{\cos \theta + \omega_\theta \sin \theta}.$$

Therefore, problem (1.1)-(2.1) is converted into the following problem

$$(2.15) \quad \begin{cases} \omega_s = a \left(\frac{\omega_\theta \cos \theta - \sin \theta}{\cos \theta + \omega_\theta \sin \theta} \right) \frac{\omega_{\theta\theta} - \omega_\theta^2 - 1}{e^{2\omega} (\cos \theta + \omega_\theta \sin \theta)^2}, & \theta \in (-\theta_0, \theta_0), \quad s \in (0, \infty), \\ \omega_\theta(-\theta_0, s) = -h_1^0(s, \omega(-\theta_0, s)), & s \in [0, \infty), \\ \omega_\theta(\theta_0, s) = h_2^0(s, \omega(\theta_0, s)), & s \in [0, \infty), \\ \omega(\theta, 0) = \omega_0(\theta), & \theta \in [-\theta_0, \theta_0]. \end{cases}$$

where ω_0 is defined by (2.14) at $t = s = 0$ and

$$(2.16) \quad h_i^0(s, \omega) = \frac{\sin \theta_0 + k_i(s, e^\omega \cos \theta_0) \cos \theta_0}{\cos \theta_0 - k_i(s, e^\omega \cos \theta_0) \sin \theta_0} \quad (i = 1, 2).$$

Estimate (2.6) implies that

$$(2.17) \quad \sigma - \tan \beta \leq u_x = \frac{\omega_\theta \cos \theta - \sin \theta}{\cos \theta + \omega_\theta \sin \theta} \leq \tan \beta - \sigma \quad \text{for } \theta \in [-\theta_0, \theta_0], \quad s \geq 0.$$

Thus $\cos \theta + \omega_\theta \sin \theta > 0$ since it can not be zero by (2.17) and it is positive at $\theta = 0$. Considering the second inequality in (2.17) we have

$$(2.18) \quad \omega_\theta [\cos \theta - \sin \theta (\tan \beta - \sigma)] \leq \sin \theta + (\tan \beta - \sigma) \cos \theta.$$

Note that, for $\theta \in [-\theta_0, \theta_0]$ ($\theta_0 = \frac{\pi}{2} - \beta$), we have

$$\cos \theta - \sin \theta (\tan \beta - \sigma) \geq \cos \theta_0 [1 - \tan \theta_0 (\tan \beta - \sigma)] \geq \sigma \cos \beta,$$

$$\cos \theta - \sin \theta (\tan \beta - \sigma) \leq \cos \theta [1 + \tan \theta_0 (\tan \beta - \sigma)] \leq 2 - \sigma \cot \beta.$$

So

$$\omega_\theta \leq \frac{\sin \theta + (\tan \beta - \sigma) \cos \theta}{\cos \theta - \sin \theta (\tan \beta - \sigma)} \leq \Omega_1 := \frac{1 + \tan \beta - \sigma}{\sigma \cos \beta}.$$

Using the first inequality in this formula we have, for $\theta \leq 0$,

$$\cos \theta + \omega_\theta \sin \theta \geq \frac{1}{\cos \theta - \sin \theta (\tan \beta - \sigma)} \geq \varepsilon_1 := \frac{1}{2 - \sigma \cot \beta}.$$

Similarly, considering the first inequality in (2.17) we have

$$\omega_\theta \geq \frac{\sin \theta - (\tan \beta - \sigma) \cos \theta}{\cos \theta + \sin \theta (\tan \beta - \sigma)} \geq -\Omega_1,$$

and $\cos \theta + v_\theta \sin \theta \geq \varepsilon_1$ for $\theta \geq 0$.

Summarizing the above results we have

$$(2.19) \quad |\omega_\theta(\theta, s)| \leq \Omega_1 \quad \text{and} \quad \cos \theta + \omega_\theta \sin \theta \geq \varepsilon_1 > 0$$

for $\theta \in [-\theta_0, \theta_0]$ and $s \geq 0$.

By the standard theory for parabolic equations, we see that (2.15) has a classical solution on time interval $s \in [0, 2\tau]$ for positive $\tau = \tau(k_1, k_2, \mu, \omega_0)$.

The second inequality in (2.19) implies that, once the solution ω of (2.15) is obtained then we can recover it to the original solution u of (1.1). In fact,

$$\frac{\partial}{\partial \theta} X(\theta, \omega(\theta, s), s) = \frac{\partial}{\partial \theta} e^{\omega(\theta, s)} \sin \theta = e^{\omega(\theta, s)} (\cos \theta + \omega_\theta \sin \theta) > 0.$$

Consequently, we have the following local existence result.

Lemma 2.3 *Problem (1.1) with initial datum $u(x, 0) = u_0(x)$ satisfying (2.2)-(2.3) has a classical solution u on time interval $[0, 2\tau]$, where τ depends only on k_1, k_2, μ and u_0 .*

3 Expanding selfsimilar solutions

In this section we always assume that (1.7) holds.

3.1 Classical expanding selfsimilar solutions

First, recall the classical selfsimilar solutions of (1.2). For any $\gamma_1, \gamma_2 \in \mathbb{R}$, consider the problem

$$(3.1) \quad \begin{cases} a(\varphi'(z))\varphi''(z) = \varphi(z) - z\varphi'(z), & z \in \mathbb{R}, \\ \varphi'(-p_1) = -\gamma_1, \quad \varphi(-p_1) = p_1 \tan \beta, \\ \varphi'(p_2) = \gamma_2, \quad \varphi(p_2) = p_2 \tan \beta. \end{cases}$$

[2, 3, 11] obtained the following result.

Lemma 3.1 *For any given γ_1, γ_2 with $\gamma_1 + \gamma_2 > 0$, there exists a unique pair $p_1, p_2 > 0$ such that problem (3.1) has a solution $\varphi(z; \gamma_1, \gamma_2)$, which is positive on $[-p_1, p_2]$.*

It is easily seen that the function

$$\sqrt{2t} \varphi \left(\frac{x}{\sqrt{2t}}; \gamma_1, \gamma_2 \right) \quad \text{for} \quad -\zeta_1(t) < x < \zeta_2(t), \quad t > 0,$$

with $\zeta_i(t) = p_i\sqrt{2t}$ ($i = 1, 2$) is an expanding selfsimilar solution of (1.2). Define

$$\varphi^-(z) := \varphi(z; k_1^0, k_2^0) \quad \text{and} \quad \varphi^+(z) := \varphi(z; K_1^0, K_2^0).$$

Then by comparison principle Lemma 2.1, $\sqrt{2t} \varphi^-(x/\sqrt{2t})$ and $\sqrt{2t} \varphi^+(x/\sqrt{2t})$ (both are expanding selfsimilar solutions of (1.2)) are lower and upper solutions of (1.1), respectively.

Since the initial datum $u_0 > 0$, there exist $t^+, t^- > 0$ such that

$$(3.2) \quad \sqrt{2t^-} \varphi^- \left(\frac{x}{\sqrt{2t^-}} \right) \leq u_0(x) \leq \sqrt{2t^+} \varphi^+ \left(\frac{x}{\sqrt{2t^+}} \right).$$

The comparison principle implies that

$$(3.3) \quad \sqrt{2(t+t^-)} \varphi^- \left(\frac{x}{\sqrt{2(t+t^-)}} \right) \leq u(x, t) \leq \sqrt{2(t+t^+)} \varphi^+ \left(\frac{x}{\sqrt{2(t+t^+)}} \right).$$

3.2 Changes of variables

In Subsection 2.5 we gave a local existence result. One difficulty for deriving the global existence is the lack of the growth bound for ω . To give the global existence we adopt another change of variables.

Let τ be the constant in Lemma 2.3 and let $t^- > 0$ be as in (3.2). For any $n \in \mathbb{N}$ satisfying

$$(3.4) \quad n > \frac{1}{t^-} \quad \text{and} \quad n > \frac{1}{\tau},$$

we introduce new variables by

$$(3.5) \quad \begin{cases} \theta = \arctan \frac{x}{y}, & (x, y) \in \overline{S} \setminus \{0\}, \\ \rho = \frac{1}{2} \log \frac{n(x^2 + y^2)}{nt + 1}, & (x, y) \in \overline{S} \setminus \{0\}, t \geq 0, \\ s = \frac{1}{2} \log \left(t + \frac{1}{n} \right), & t \geq 0. \end{cases}$$

The inverse map is

$$(3.6) \quad \begin{cases} x = e^s e^\rho \sin \theta, & (\theta, \rho) \in \overline{D}, s \in [-\frac{1}{2} \log n, \infty), \\ y = e^s e^\rho \cos \theta, & (\theta, \rho) \in \overline{D}, s \in [-\frac{1}{2} \log n, \infty), \\ t = e^{2s} - \frac{1}{n}, & s \in [-\frac{1}{2} \log n, \infty). \end{cases}$$

Clearly, $\theta = \theta_0$ and $\theta = -\theta_0$ correspond to $\partial_2 S$ and $\partial_1 S$, respectively.

Let $u(x, t) > 0$ be a solution of (1.1)-(2.1) for $t \geq 0$, then a similar discussion as in Subsection 2.5 shows that

$$(3.7) \quad \rho(x, u(x, t), t) = v(\theta(x, u(x, t), t), s(t)) \quad \Leftrightarrow \quad e^s e^{v(\theta, s)} \cos \theta = u \left(e^s e^{v(\theta, s)} \sin \theta, e^{2s} - \frac{1}{n} \right)$$

defines a new unknown $\rho = v(\theta, s)$ for $s \in [-\frac{1}{2} \log n, \infty)$. Differentiating the second equality twice by θ and once by t we obtain

$$u_x = \frac{v_\theta \cos \theta - \sin \theta}{\cos \theta + v_\theta \sin \theta}, \quad u_{xx} = \frac{v_{\theta\theta} - v_\theta^2 - 1}{e^s e^v (\cos \theta + v_\theta \sin \theta)^3}, \quad u_t = \frac{e^v (1 + v_s)}{2e^s (\cos \theta + v_\theta \sin \theta)}.$$

Therefore, problem (1.1) is converted into the following problem

$$(3.8) \quad \begin{cases} v_s = 2a \left(\frac{v_\theta \cos \theta - \sin \theta}{\cos \theta + v_\theta \sin \theta} \right) \frac{v_{\theta\theta} - v_\theta^2 - 1}{e^{2v} (\cos \theta + v_\theta \sin \theta)^2} - 1, & -\theta_0 < \theta < \theta_0, \quad s > -\frac{1}{2} \log n, \\ v_\theta(-\theta_0, s) = -g_1(s, v(-\theta_0, s)), & s \geq -\frac{1}{2} \log n, \\ v_\theta(\theta_0, s) = g_2(s, v(\theta_0, s)), & s \geq -\frac{1}{2} \log n, \end{cases}$$

where

$$(3.9) \quad g_i(s, v) = \frac{\sin \theta_0 + k_i \left(e^{2s} - \frac{1}{n}, e^s e^v \cos \theta_0 \right) \cos \theta_0}{\cos \theta_0 - k_i \left(e^{2s} - \frac{1}{n}, e^s e^v \cos \theta_0 \right) \sin \theta_0} \quad (i = 1, 2).$$

3.3 Gradient bound of v

In a similar way as deriving (2.19) in Subsection 2.5 one can obtain

$$(3.10) \quad |v_\theta(\theta, s)| \leq \Omega_2(\sigma, \beta) \quad \text{and} \quad \cos \theta + v_\theta \sin \theta \geq \varepsilon_2(\sigma, \beta) > 0$$

for $\theta \in [-\theta_0, \theta_0]$ and $s \in [-\frac{1}{2} \log n, \infty)$.

The second inequality in (3.10) implies that, once the solution v of (3.8) is obtained then we can recover it to the original solution u of (1.1), since

$$\frac{\partial}{\partial \theta} X(\theta, v(\theta, s), s) = \frac{\partial}{\partial \theta} e^s e^{v(\theta, s)} \sin \theta = e^s e^{v(\theta, s)} (\cos \theta + v_\theta \sin \theta) > 0.$$

3.4 Bound of v

The local existence result Lemma 2.3 implies that v exists on $s \in [-\frac{1}{2} \log n, \frac{1}{2} \log(2\tau + \frac{1}{n})]$. We have changed $u(x, t)$ to a new unknown $v(\theta, s)$. Similarly, we define v^\pm by φ^\pm in the following way

$$e^s e^{v^\pm} \cos \theta = \sqrt{2 \left(e^{2s} - \frac{1}{n} + t^\pm \right)} \varphi^\pm \left(\frac{e^s e^{v^\pm} \sin \theta}{\sqrt{2 \left(e^{2s} - \frac{1}{n} + t^\pm \right)}} \right).$$

By (3.3) we have $e^s e^{v^+} \cos \theta \geq u(e^s e^{v^+} \sin \theta, e^{2s} - \frac{1}{n})$. Noting $e^s e^v \cos \theta = u(e^s e^v \sin \theta, e^{2s} - \frac{1}{n})$ we have

$$(e^{v^+} - e^v) \cos \theta \geq u_x\left(\vartheta, e^{2s} - \frac{1}{n}\right) \cdot [(e^{v^+} - e^v) \sin \theta],$$

where $\vartheta = e^s \sin \theta (\varsigma e^v + (1 - \varsigma) e^{v^+})$ for some $\varsigma \in [0, 1]$. Therefore $v^+ \geq v$ by (2.7).

On the other hand, by the definition of v^+ we have

$$(3.11) \quad e^{v^+} \cos \theta = \varphi^+(\cdot) \sqrt{2 + \left(t^+ - \frac{1}{n}\right) e^{-2s}} \leq \sqrt{2 + t^+/\tau} \max \varphi^+ \quad \text{for } s \geq \frac{1}{2} \log \tau.$$

So

$$v(\theta, s) \leq v^+(\theta, s) \leq \log \left[\frac{\sqrt{2 + t^+/\tau} \max \varphi^+}{\sin \beta} \right] \quad \text{for } \theta \in [-\theta_0, \theta_0], \quad s \geq \frac{1}{2} \log \tau.$$

A similar discussion as above shows that

$$v(\theta, s) \geq v^-(\theta, s) \geq \log [2 \min \varphi^-] \quad \text{for } \theta \in [-\theta_0, \theta_0], \quad s \geq \frac{1}{2} \log \tau.$$

Remark 3.1 *The definition of v depends on n , it is not easy to give a uniform (for n) bound for v on $[-\frac{1}{2} \log n, \infty)$, but the above results show that a uniform (for n) bound for v is possible on $[\frac{1}{2} \log \tau, \infty)$.*

3.5 Global existence

Now we consider problem (3.8) with initial datum $v(\theta, -\frac{1}{2} \log n) = v_0(\theta)$, which is defined by (3.7) at $s = -\frac{1}{2} \log n$. Using the bound and gradient bound in the previous subsections and using the standard theory of parabolic equations (cf. [5, 6, 12, 13]) we can get the following conclusions.

Lemma 3.2 *Problem (3.8) with initial datum $v(\theta, -\frac{1}{2} \log n) = v_0(\theta)$ has a unique, time-global solution $v(\theta, s) \in C^{2+\mu, 1+\mu/2}([-\theta_0, \theta_0] \times [-\frac{1}{2} \log n, \infty))$ and*

$$(3.12) \quad \|v(\theta, s)\|_{C^{2+\mu, 1+\mu/2}([-\theta_0, \theta_0] \times [\frac{1}{2} \log \tau, \infty))} \leq C_0 < \infty,$$

where C_0 depends on k_1, k_2, μ and u_0 but not on s and n .

This lemma implies the global existence of u .

Lemma 3.3 *Problem (1.1)-(2.1) has a unique, time-global solution $u(x, t)$. Moreover, $u \in C^{2+\mu, 1+\mu/2}(\overline{Q_\infty})$, where $Q_\infty := \{(x, t) \mid -\xi_1(t) < x < \xi_2(t), t > 0\}$. For any $t_0 > \tau$,*

$$\|u(x, t)\|_{C^{2+\mu, 1+\mu/2}(\overline{Q_{t_0}} \setminus Q_\tau)} \leq C_1(t_0, k_1, k_2, \mu, u_0, \tau) < \infty.$$

Indeed, studying the relations between v and u more precisely, it is not difficult to see that C_1 in this lemma can be replaced by $C_2\sqrt{t_0} + C_3$ for some C_2, C_3 depending on k_1, k_2, μ, u_0 and τ .

3.6 Existence of selfsimilar solution

Since the solution v in Lemma 3.2 is defined for $s \geq -\frac{1}{2}\log n$, we write it as v_n . By Cantor's diagonal argument, one can find a function $V \in C^{2+\mu, 1+\mu/2}([- \theta_0, \theta_0] \times [\frac{1}{2}\log \tau, \infty))$ and a subsequence $\{n_i\} \subset \{n\}$ such that, as $i \rightarrow \infty$,

$$(3.13) \quad v_{n_i}(\theta, s) \rightarrow V(\theta, s) \quad \text{in } C_{\text{loc}}^{2,1}([- \theta_0, \theta_0] \times [\frac{1}{2}\log \tau, \infty)) \text{ topology.}$$

Moreover, V satisfies the estimate

$$(3.14) \quad \|V(\theta, s)\|_{C^{2+\mu, 1+\mu/2}([- \theta_0, \theta_0] \times [\frac{1}{2}\log \tau, \infty))} \leq C_0 < \infty,$$

and V is a solution of

$$(3.15) \quad \begin{cases} v_s = 2a \left(\frac{v_\theta \cos \theta - \sin \theta}{\cos \theta + v_\theta \sin \theta} \right) \frac{v_{\theta\theta} - v_\theta^2 - 1}{e^{2v}(\cos \theta + v_\theta \sin \theta)^2} - 1, & -\theta_0 < \theta < \theta_0, \ s \geq \frac{1}{2}\log \tau, \\ v_\theta(-\theta_0, s) = -G_1(s, v(-\theta_0, s)), & s \geq \frac{1}{2}\log \tau, \\ v_\theta(\theta_0, s) = G_2(s, v(\theta_0, s)), & s \geq \frac{1}{2}\log \tau, \end{cases}$$

where

$$(3.16) \quad G_i(s, v) = \frac{\sin \theta_0 + k_i \left(e^{2s}, e^s e^v \cos \theta_0 \right) \cos \theta_0}{\cos \theta_0 - k_i \left(e^{2s}, e^s e^v \cos \theta_0 \right) \sin \theta_0} \quad (i = 1, 2).$$

Here G_1 and G_2 are $\log b$ -periodic functions in s by (1.5).

For any $n \in \mathbb{N}$, $V_n(\theta, s) := V(\theta, s + n \log b)$ is a solution of (3.15) for $s \geq \frac{1}{2}\log T - n \log b$, and

$$\|V_n(\theta, s)\|_{C^{2+\mu, 1+\mu/2}([- \theta_0, \theta_0] \times [\frac{1}{2}\log T - n \log b, \infty))} \leq C_0$$

by (3.14). Using Cantor's diagonal argument, one can find a function $P \in C^{2+\mu, 1+\mu/2}([- \theta_0, \theta_0] \times \mathbb{R})$ and a subsequence $\{n_i\} \subset \{n\}$ such that, as $i \rightarrow \infty$,

$$(3.17) \quad V_{n_i}(\theta, s) \rightarrow P(\theta, s) \quad \text{in } C_{\text{loc}}^{2,1}([- \theta_0, \theta_0] \times \mathbb{R}) \text{ topology.}$$

Moreover, P is a solution of (3.15) for $s \in \mathbb{R}$ and it satisfies the estimate

$$(3.18) \quad \|P(\theta, s)\|_{C^{2+\mu, 1+\mu/2}([- \theta_0, \theta_0] \times \mathbb{R})} \leq C_0.$$

Now we use a result in [1].

Lemma 3.4 *Let $u(\cdot, t)$ be a bounded (in $H^2(0, 1)$) solution of*

$$(3.19) \quad \begin{cases} u_t = d(t, x, u, u_x)u_{xx} + f(t, x, u, u_x), & t > 0, 0 < x < 1, \\ u_x(i, t) = g_i^*(t, u(i, t)), & i = 0, 1, t > 0, \end{cases}$$

where d, f, g_i^* are C^2 functions, T -periodic in t . Then there exists a T -periodic solution $p(x, t)$ of (3.19) such that $\lim_{t \rightarrow \infty} \|u(\cdot, t) - p(\cdot, t)\|_{H^2} = 0$.

This lemma implies that the limit P of V_{n_i} is a $\log b$ -time-periodic solution of (3.15).

We now recover P to a solution of (1.1), the corresponding change of variables should be the limiting version as $n \rightarrow \infty$ of (3.5) and (3.6), or for $s \in (-\infty, \infty)$. Using these variables we define U by

$$e^s e^P \cos \theta = U(e^s e^P \sin \theta, e^{2s}) \quad \text{for } \theta \in [-\theta_0, \theta_0], s \in (-\infty, \infty).$$

By (3.10), (3.13) and (3.17) we have

$$\frac{\partial}{\partial \theta} X(\theta, P(\theta, s), s) = \frac{\partial}{\partial \theta} (e^s e^{P(\theta, s)} \sin \theta) = e^s e^{P(\theta, s)} (P_\theta \sin \theta + \cos \theta) > 0,$$

so the function $U(x, t)$ is well-defined for all $t > 0$. Moreover, by the definition of U and the periodicity of P we have

$$\begin{aligned} bU(e^s e^{P(\theta, s)} \sin \theta, e^{2s}) &= be^s e^{P(\theta, s)} \cos \theta = e^{s+\log b} e^{P(\theta, s+\log b)} \cos \theta \\ &= U(e^{s+\log b} e^{P(\theta, s+\log b)} \sin \theta, e^{2s+2\log b}) \\ &= U(be^s e^{P(\theta, s)} \sin \theta, b^2 e^{2s}). \end{aligned}$$

Hence

$$bU(x, t) = U(bx, b^2 t) \quad \text{for } -\Xi_1(t) \leq x \leq \Xi_2(t), t > 0,$$

where $-\Xi_1(t)$ and $\Xi_2(t)$ are the x -coordinate of the end points of the graph of U . Since $t = e^{2s}$ we have

$$\Xi_2(t) = e^s e^{P(\theta_0, s)} \sin \theta_0 = \sqrt{t} e^{P(\theta_0, \frac{1}{2} \log t)} \sin \theta_0 \quad \text{for } t > 0.$$

So

$$\Xi_2(b^2 t) = b\sqrt{t} e^{P(\theta_0, \frac{1}{2} \log t + \log b)} \sin \theta_0 = b\sqrt{t} e^{P(\theta_0, \frac{1}{2} \log t)} \sin \theta_0 = b\Xi_2(t) \quad \text{for } t > 0.$$

Similarly we have $\Xi_1(b^2 t) = b\Xi_1(t)$ for $t > 0$. Consequently, we obtain a selfsimilar solution of (1.1) and this proves the existence part of Theorem 1.1. \square

3.7 Uniqueness of selfsimilar solutions

In this subsection we assume $k_i(t, u) \equiv k_i(u)$ ($i = 1, 2$) and to prove the uniqueness conclusion in Theorem 1.1. The uniqueness for general $k_i(t, u)$ is still open.

We begin with choosing a convex initial datum u_0 , that is,

$$a(u_{0x})u_{0xx} \geq \epsilon$$

for some $\epsilon > 0$. Such a choice is possible. For example, draw a line ℓ_1 from $A_1 := (-1, \tan \beta)$ with slope $-k_1(0, \tan \beta)$. Denote the contacting point between ℓ_1 and $\partial_2 S$ by A'_2 , then $A'_2 := (x', x' \tan \beta)$ with

$$x' := \frac{\tan \beta - k_1(0, \tan \beta)}{\tan \beta + k_1(0, \tan \beta)}.$$

Choose $A_2 := (x' + \epsilon', (x' + \epsilon') \tan \beta) \in \partial_2 S$ for some small $\epsilon' > 0$, then A_2 is above A'_2 . Draw a line ℓ_2 from A_2 with slope $k_2(0, (x' + \epsilon') \tan \beta)$. Since

$$k_2(0, (x' + \epsilon') \tan \beta) \geq k_2^0 > -k_1^0 \geq -k_1(0, \tan \beta),$$

ℓ_2 must contact ℓ_1 at some point $A_3 \in S$ provided $\epsilon' > 0$ is small enough. Now we smoothen $\overline{A_1 A_3} + \overline{A_3 A_2}$ such that the smoothened curve \mathcal{C} is strictly convex, it is tangent to $\overline{A_1 A_3}$ at A_1 , tangent to $\overline{A_3 A_2}$ at A_2 . Now the corresponding function u_0 of \mathcal{C} is a desired initial datum.

Let u be the solution of (1.1) with above constructed initial datum u_0 . Denote $\eta = u_t$. Differentiating the problem (1.1) w.r.t t we have

$$\begin{cases} \eta_t = a(u_x)\eta_{xx} + a'(u_x)u_{xx}\eta_x, & -\xi_1(t) < x < \xi_2(t), \quad t > 0, \\ \eta_x(-\xi_1(t), t) = f_1(t)\eta(-\xi_1(t), t), \quad \eta_x(\xi_2(t), t) = f_2(t)\eta(\xi_2(t), t), & t > 0, \\ \eta(x, 0) = a(u_{0x})u_{0xx} \geq \epsilon > 0, \end{cases}$$

where f_1 and f_2 are continuous functions. Maximum principle implies that $\eta = u_t > 0$ for $t > 0$. Using the same notions as in previous subsections we have $1 + v_s > 0$ and so $1 + V_s \geq 0$ for $s \geq \frac{1}{2} \log \tau$. Using (3.17) one has $1 + P_s \geq 0$ for all $s \in \mathbb{R}$, this implies that $U_t \geq 0$. Finally, the strong maximum principle implies that $U_t > 0$ for all $t > 0$.

Assume that, besides (U, Ξ_1, Ξ_2) , there is another selfsimilar solution (U^*, Ξ_1^*, Ξ_2^*) such that

$$U^*(\bar{x}, \bar{t}) = U(\bar{x}, \bar{t}), \quad U^*(x, \bar{t}) \neq U(x, \bar{t}).$$

Set $\tilde{t} := \inf\{t \mid U^*(x, \bar{t}) < U(x, t)\}$, then

$$(3.20) \quad \tilde{t} \geq \bar{t} \quad \text{and} \quad U^*(x, \bar{t}) \leq U(x, \tilde{t}).$$

Since both U and U^* are selfsimilar we have

$$b^{-1}U^*(bx, b^2\bar{t}) \equiv U^*(x, \bar{t}) \quad \text{for } -\Xi_1^*(\bar{t}) \leq x \leq \Xi_2^*(\bar{t}),$$

$$b^{-1}U(bx, b^2\tilde{t}) \equiv U(x, \tilde{t}) \quad \text{for } -\Xi_1(\tilde{t}) \leq x \leq \Xi_2(\tilde{t}).$$

Hence (3.20) implies that

$$U^*(x, b^2\bar{t}) \preceq U(x, b^2\tilde{t}) \quad \text{for } -\Xi_1^*(b^2\bar{t}) \leq x \leq \Xi_2^*(b^2\bar{t}).$$

On the other hand, by comparison principle Lemma 2.1 and (3.20) we have

$$U^*(x, b^2\bar{t}) \equiv U^*(x, \bar{t} + (b^2 - 1)\bar{t}) < U(x, \tilde{t} + (b^2 - 1)\bar{t}) - \epsilon''$$

for some $\epsilon'' > 0$. So $b^2\tilde{t} < \tilde{t} + (b^2 - 1)\bar{t}$ since $U_t > 0$, that is, $\tilde{t} < \bar{t}$, a contradiction. Therefore, (U, Ξ_1, Ξ_2) is the unique selfsimilar solution of (1.1). This completes the proof of Theorem 1.1. \square

4 Shrinking selfsimilar solutions

In this section we always assume that (1.14) holds.

4.1 Classical shrinking/backward selfsimilar solutions

First, recall the classical selfsimilar solutions of (1.2). For any $\gamma_1, \gamma_2 \in \mathbb{R}$, consider the problem

$$(4.1) \quad \begin{cases} a(\psi'(z))\psi''(z) = z\psi'(z) - \psi(z), & z \in \mathbb{R}, \\ \psi'(-q_1) = -\gamma_1, \quad \psi(-q_1) = q_1 \tan \beta, \\ \psi'(q_2) = \gamma_2, \quad \psi(q_2) = q_2 \tan \beta. \end{cases}$$

[9, 10] obtained the following result.

Lemma 4.1 *For any γ_1, γ_2 satisfying $\gamma_1 + \gamma_2 < 0$, there exists a pair $q_1, q_2 > 0$ such that problem (4.1) has solution $\psi(z; \gamma_1, \gamma_2)$, which is positive on $[-q_1, q_2]$.*

It is easily seen that the function

$$\sqrt{2(T-t)} \psi \left(\frac{x}{\sqrt{2(T-t)}} \right) \quad \text{for } -\zeta_1(t) < x < \zeta_2(t), \quad t > 0,$$

with $\zeta_i(t) = q_i \sqrt{2(T-t)}$ ($i = 1, 2$) is a classical shrinking/backward selfsimilar solution of (1.2).

We use $(\psi^-(z), q_1^-, q_2^-)$ to denote the solution of (4.1) with $\gamma_i = k_i^0$ ($i = 1, 2$), use $(\psi^+(z), q_1^+, q_2^+)$ to denote the solution of (4.1) with $\gamma_i = K_i^0$ ($i = 1, 2$), where k_i^0 and K_i^0 are those in (2.4).

By (1.14), $K_1^0 + K_2^0 < 0$ and so $(\psi^\pm)'' < 0$. For any $T_0 > 0$, the function $\sqrt{2(T_0 - t)} \psi^-(x/\sqrt{2(T_0 - t)})$ and the function $\sqrt{2(T_0 - t)} \psi^+(x/\sqrt{2(T_0 - t)})$ (both are shrinking/backward selfsimilar solutions of (1.2)) are lower and upper solutions of (1.1), respectively. Since the initial datum $u_0 > 0$, there exists $T^+, T^- > 0$ such that

$$(4.2) \quad \sqrt{2T^-} \psi^- \left(\frac{x}{\sqrt{2T^-}} \right) \preceq u_0(x) \preceq \sqrt{2T^+} \psi^+ \left(\frac{x}{\sqrt{2T^+}} \right).$$

Comparison principle implies that

$$(4.3) \quad \sqrt{2(T^- - t)} \psi^- \left(\frac{x}{\sqrt{2(T^- - t)}} \right) \leq u(x, t) \leq \sqrt{2(T^+ - t)} \psi^+ \left(\frac{x}{\sqrt{2(T^+ - t)}} \right)$$

on the time interval where these three functions are defined.

Lemma 4.2 *Let ψ^\pm and T^\pm be as in (4.2). Then $T^+ > T^- \geq \delta T^+$, where*

$$\delta = \delta(\beta, \sigma, k_i^0, K_i^0) := \frac{\sigma^4}{(2 \tan \beta - \sigma)^4} \cdot \frac{(q_2^+)^2}{(q_2^-)^2} > 0.$$

Proof. We first prove $T^+ > T^-$. The areas $D^\pm(t)$ of the regions enclosed by the graph of $\sqrt{2(T^\pm - t)} \psi^\pm(x/\sqrt{2(T^\pm - t)})$, $\partial_1 S$ and $\partial_2 S$ are given by

$$D^\pm(t) = \int_{-\zeta_1(t)}^{\zeta_2(t)} \sqrt{2(T^\pm - t)} \psi^\pm \left(\frac{x}{\sqrt{2(T^\pm - t)}} \right) dx - \frac{1}{2} [\zeta_1^2(t) + \zeta_2^2(t)] \tan \beta.$$

A simple computation shows that

$$(D^+)'(t) = \int_{-K_1^0}^{K_2^0} a(p) dp, \quad (D^-)'(t) = \int_{-k_1^0}^{k_2^0} a(p) dp.$$

Since $D^+(T^+) = D^-(T^-) = 0$ we have

$$(4.4) \quad D^+(0) = \int_0^{T^+} dt \int_{K_2^0}^{-K_1^0} a(p) dp, \quad D^-(0) = \int_0^{T^-} dt \int_{k_2^0}^{-k_1^0} a(p) dp.$$

(1.14) implies that $k_2^0 < K_2^0 < -K_1^0 < -k_1^0$, (4.2) implies that $D^+(0) \geq D^-(0)$. Therefore, by (4.4) we have $T^+ > T^-$.

Next we prove $T^- \geq \delta T^+$. For $i = 1, 2$, denote Q_i^+ (resp. Q_i^-) the end points of the graph of $\sqrt{2T^+} \psi^+(x/\sqrt{2T^+})$ (resp. the graph of $\sqrt{2T^-} \psi^-(x/\sqrt{2T^-})$) on $\partial_i S$, respectively.

Connecting Q_1^+ and Q_2^+ we get a line segment $\overline{Q_1^+ Q_2^+}$. It is below the graph of $\sqrt{2T^+} \psi^+(x/\sqrt{2T^+})$ since $(\psi^+)'' < 0$. Draw a line from Q_1^+ (resp. Q_2^+) with slope $-\tan \beta + \sigma$ (resp. $\tan \beta - \sigma$). Assume that it contacts $\partial_2 S$ (resp. $\partial_1 S$) at A_2 (resp. A_1).

By (4.2) the graph of u_0 contacts the line segment $\overline{Q_1^+ Q_2^+}$. Since $|u_{0x}| \leq \tan \beta - \sigma$, we see that the graph of u_0 is above the line segment $\overline{A_1 A_2}$.

If for some $T_0 > 0$, the graph of $\sqrt{2T_0} \psi^-(x/\sqrt{2T_0})$ is tangent to $\overline{A_1 A_2}$ from above, then (note $|(\psi^-)'| \leq \tan \beta - \sigma$) the graph of $\sqrt{2T_0} \psi^-(x/\sqrt{2T_0})$ lies above the line segment $\overline{B_1 B_2}$, where B_1 (resp. B_2) is the contacting point between the line passing A_2 (resp. A_1) with slope $\tan \beta - \sigma$ (resp. $-\tan \beta + \sigma$) and the left boundary $\partial_1 S$ (resp. right boundary $\partial_2 S$). Therefore, Q_i^- are above B_i for $i = 1, 2$.

Using the coordinates of $Q_i^+ = ((-1)^i r_i \cos \beta, r_i \sin \beta)$, where

$$r_i = \frac{\sqrt{2T^+} q_i^+}{\cos \beta} \quad (i = 1, 2),$$

one can easily calculate the coordinates of B_1 and B_2 :

$$B_i = \left(\frac{(-1)^i \sigma^2 r_i \cos \beta}{(2 \tan \beta - \sigma)^2}, \frac{\sigma^2 r_i \sin \beta}{(2 \tan \beta - \sigma)^2} \right) \quad (i = 1, 2).$$

The fact that Q_2^- is above B_2 implies that

$$\sqrt{2T^-} q_2^- \geq \frac{\sigma^2 \cos \beta}{(2 \tan \beta - \sigma)^2} \cdot \frac{\sqrt{2T^+} q_2^+}{\cos \beta}.$$

So

$$\frac{T^-}{T^+} \geq \left[\frac{\sigma^2 q_2^+}{q_2^- (2 \tan \beta - \sigma)^2} \right]^2.$$

This proves the lemma. □

4.2 Shrinking time for solutions of (1.1)-(2.1)

In this subsection we consider the shrinking time for the solution $u = u(x, t; u_0)$ of (1.1) with initial datum u_0 . We give two results. The first one is about the shrinking time of $u = u(x, t; u_0)$ for given u_0 , the second one is about the existence of u_0 for given shrinking time T .

Lemma 4.3 *Let u_0 be an initial datum satisfying (4.2) and all the conditions in Subsection 2.1. Then there exists $T \in (T^-, T^+)$ such that the solution $u(x, t; u_0)$ of (1.1)-(2.1) exists on time interval $[0, T)$, and*

$$(4.5) \quad \|u(\cdot, t)\|_{L^\infty} \rightarrow 0, \quad \xi_1(t) \rightarrow 0, \quad \xi_2(t) \rightarrow 0 \quad \text{as } t \rightarrow T.$$

Proof. We use polar coordinates $x = r \sin \theta$, $y = r \cos \theta$ for $\theta \in [-\theta_0, \theta_0]$ to define a new function $r = r(\theta, t)$ for $t \geq 0$:

$$r \cos \theta = u(r \sin \theta, t).$$

This formula indeed defines an explicit function $r = r(\theta, t)$ by (2.7) (A similar discussion as in Subsection 2.5 also shows that r is well-defined). Problem (1.1) is then converted into

$$(4.6) \quad \begin{cases} r_t = a \left(\frac{r_\theta \cos \theta - r \sin \theta}{r_\theta \sin \theta + r \cos \theta} \right) \frac{rr_{\theta\theta} - 2r_\theta^2 - r^2}{r(r_\theta \sin \theta + r \cos \theta)^2}, & \theta \in [-\theta_0, \theta_0], \quad t > 0, \\ r_\theta(-\theta_0, t) = -\tilde{h}_1(t, r(-\theta_0, t)), & t \geq 0, \\ r_\theta(\theta_0, t) = \tilde{h}_2(t, r(\theta_0, t)), & t \geq 0, \end{cases}$$

where

$$\tilde{h}_i(t, r) := \frac{\sin \theta_0 + k_i(t, r \cos \theta_0) \cos \theta_0}{\cos \theta_0 - k_i(t, r \cos \theta_0) \sin \theta_0} r \quad (i = 1, 2).$$

By (4.3) we have $r \cos \theta = u(r \sin \theta, t) \leq \sqrt{2T^+} \max \psi^+$. So $0 \leq r(\theta, t) \leq \sqrt{2T^+} \max \psi^+ / \sin \beta$. By (2.6) and (2.7) we have

$$|r_\theta| = \left| \frac{\sin \theta + u_x \cos \theta}{\cos \theta - u_x \sin \theta} r \right| \leq \frac{1 + \tan \beta - \sigma}{\sigma \cos \beta} \cdot \frac{\sqrt{2T^+} \max \psi^+}{\sin \beta}.$$

Thus the standard a priori estimates (cf. [5, 6, 12, 13]) show that the solution of (4.6) with initial datum $r(\theta, 0)$ will not develop singularity till $\min r(\cdot, t) \rightarrow 0$ as $t \rightarrow T$ for some $T > 0$. Moreover, $T \in (T^-, T^+)$ by (4.3) and Lemma 4.2.

Now we prove (4.5). If $u(0, T+0) = 0$ but $u(\bar{x}, T+0) > 0$. Without loss of generality we assume $\bar{x} > 0$, then there exists $\hat{x} \in (0, \bar{x})$ such that

$$u_x(\hat{x}, T+0) \geq \frac{u(\bar{x}, T+0) - u(0, T+0)}{\bar{x}} \geq \tan \beta.$$

This contradicts Lemma 2.2 and so the first limit in (4.5) holds. The last two limits in (4.5) follow from the first one. This proves the lemma. \square

Lemma 4.4 *For any given $T > 0$, there exists an initial datum u_0 such that the solution $u(x, t; u_0)$ of (1.1)-(2.1) exists on $[0, T)$ and shrinks to 0 just at time T .*

Proof. Choose two initial datum u_0^\pm such that they satisfy conditions (2.2) and (2.3). Moreover, we choose u_0^+ large such that (4.2) holds for some $T^- > T$. By Lemma 4.3, the solution $u(x, t; u_0^+)$ shrinks to 0 as $t \rightarrow \tilde{T}^+$ for some $\tilde{T}^+ > T^- > T$. On the other hand, we choose u_0^- small such that (4.2) holds for some $T^+ < T$. By Lemma 4.3 again, the solution $u(x, t; u_0^-)$ shrinks to 0 as $t \rightarrow \tilde{T}^-$ for some $\tilde{T}^- < T^+ < T$.

Now we modify the initial datum from u_0^- to u_0^+ little by little such that the modified initial datum still satisfies (2.2) and (2.3). Since the solution $u(x, t; u_0)$ of (1.1)-(2.1) depends on the initial datum u_0 continuously, we finally have an initial datum u_0 such that $u(x, t; u_0)$ shrinks to 0 at time $T \in (\tilde{T}^-, \tilde{T}^+)$. \square

In the following of this section, we fix $T > 0$ and choose the initial datum as in Lemma 4.4.

4.3 Change of variables

Lemma 4.3 gives the existence and boundedness of r (and so, of u), but the time interval is finite: $[0, T)$. So Lemma 3.4 can not be applied to give a periodic solution. To get a shrinking selfsimilar solution of (1.1), we introduce new coordinates as above. Set

$$(4.7) \quad \begin{cases} \theta = \arctan \frac{x}{y}, & (x, y) \in \overline{S} \setminus \{0\}, \\ \rho = -\frac{1}{2} \log \frac{x^2 + y^2}{T - t}, & (x, y) \in \overline{S} \setminus \{0\}, \quad 0 \leq t < T, \\ s = -\frac{1}{2} \log(T - t), & 0 \leq t < T. \end{cases}$$

The inverse map is

$$(4.8) \quad \begin{cases} x = e^{-s} e^{-\rho} \sin \theta, & (\theta, \rho) \in \overline{D}, \quad s \in [-\frac{1}{2} \log T, \infty), \\ y = e^{-s} e^{-\rho} \cos \theta, & (\theta, \rho) \in \overline{D}, \quad s \in [-\frac{1}{2} \log T, \infty), \\ t = T - e^{-2s}, & s \in [-\frac{1}{2} \log T, \infty). \end{cases}$$

A similar discussion as in Subsections 2.5 and 3.2 shows that in these new variables, the original function $y = u(x, t)$ is converted into a new function $\rho = w(\theta, s)$. Differentiating the expression

$$(4.9) \quad e^{-s} e^{-w(\theta, s)} \cos \theta = u(e^{-s} e^{-w(\theta, s)} \sin \theta, T - e^{-2s})$$

twice by θ and once by s we obtain

$$(4.10) \quad u_x = \frac{w_\theta \cos \theta + \sin \theta}{w_\theta \sin \theta - \cos \theta}, \quad u_{xx} = \frac{e^s e^w (w_{\theta\theta} + w_\theta^2 + 1)}{(w_\theta \sin \theta - \cos \theta)^3}, \quad u_t = \frac{e^s (1 + w_s)}{2e^w (w_\theta \sin \theta - \cos \theta)}.$$

Therefore, problem (1.1) is converted into the following problem

$$(4.11) \quad \begin{cases} w_s = 2e^{2w}a\left(\frac{w_\theta \cos \theta + \sin \theta}{w_\theta \sin \theta - \cos \theta}\right) \frac{w_{\theta\theta} + w_\theta^2 + 1}{(w_\theta \sin \theta - \cos \theta)^2} - 1, \\ -\theta_0 < \theta < \theta_0, \quad s \in [-\frac{1}{2} \log T, \infty), \\ w_\theta(-\theta_0, s) = \tilde{g}_1(s, w(-\theta_0, s)), \quad s \in [-\frac{1}{2} \log T, \infty), \\ w_\theta(\theta_0, s) = -\tilde{g}_2(s, w(\theta_0, s)), \quad s \in [-\frac{1}{2} \log T, \infty), \end{cases}$$

where

$$(4.12) \quad \tilde{g}_i(s, w) = \frac{\sin \theta_0 + k_i(T - e^{-2s}, e^{-s}e^{-w} \cos \theta_0) \cos \theta_0}{\cos \theta_0 - k_i(T - e^{-2s}, e^{-s}e^{-w} \cos \theta_0) \sin \theta_0} \quad (i = 1, 2).$$

4.4 Bound of w

We derive the boundedness of w in a series time intervals: $[0, \delta^2 T]$, $[\delta^2 T, T - (1 - \delta^2)^2 T]$, $[T - (1 - \delta^2)^2 T, T - (1 - \delta^2)^3 T]$, \dots , where δ is that in Lemma 4.2.

In the first step, we choose ψ^\pm and T^\pm as in (4.2), and consider (1.1) on time interval $t \in [0, \delta^2 T]$ (note that $\delta^2 T < \delta^2 T^+ \leq \delta T^- < \delta T < \delta T^+ \leq T^-$), or, equivalently, consider (4.11) on time-interval $s \in [-\frac{1}{2} \log T, -\frac{1}{2} \log T - \frac{1}{2} \log(1 - \delta^2)]$. In this period,

$$e^{-2s} = T - t \geq (1 - \delta^2)T \quad \Rightarrow \quad e^{2s}T \leq \frac{1}{1 - \delta^2}.$$

Thus,

$$\begin{aligned} e^{2s}(T^+ - t) &\leq e^{2s}[(T^+ - T^-) + (T - t)] \leq e^{2s} \left[\frac{1 - \delta}{\delta} T^- + (T - t) \right] \\ &\leq \frac{1 - \delta}{\delta} e^{2s}T + 1 \leq \delta_1 := 1 + \frac{1}{\delta(1 + \delta)}. \end{aligned}$$

By (4.3) we have, for $t \in [0, \delta^2 T]$ or $s \in [-\frac{1}{2} \log T, -\frac{1}{2} \log T - \frac{1}{2} \log(1 - \delta^2)]$,

$$e^{-s}e^{-w} \cos \theta = u(\cdot, T - e^{-2s}) \leq \sqrt{2(T^+ - t)} \max \psi^+.$$

So

$$e^{-w} \leq \frac{\max \psi^+}{\cos \theta_0} e^s \sqrt{2(T^+ - t)} \leq \frac{\max \psi^+}{\cos \theta_0} \sqrt{2\delta_1}.$$

On the other hand, in the same time interval $t \in [0, \delta^2 T]$ we have

$$\frac{T - T^-}{T - t} \leq \frac{T - T^-}{T - \delta^2 T} \leq \frac{T - \delta T}{T - \delta^2 T} = \frac{1}{1 + \delta}.$$

So

$$e^{2s}(T^- - t) = \frac{(T^- - T) + (T - t)}{T - t} \geq 1 - \frac{1}{1 + \delta} = \frac{\delta}{1 + \delta}.$$

Thus by

$$e^{-s}e^{-w} \cos \theta = u(\cdot, T - e^{-2s}) \geq \sqrt{2(T^- - t)} \min \psi^-,$$

we have

$$e^{-w} \geq \min \psi^- e^s \sqrt{2(T^- - t)} \geq \min \psi^- \sqrt{\frac{2\delta}{1+\delta}}.$$

Therefore we obtain the bound of w for $s \in [-\frac{1}{2} \log T, -\frac{1}{2} \log T - \frac{1}{2} \log(1 - \delta^2)]$:

$$(4.13) \quad -\log \left[\frac{\max \psi^+}{\cos \theta_0} \sqrt{2\delta_1} \right] \leq w \leq -\log \left[\min \psi^- \sqrt{\frac{2\delta}{1+\delta}} \right].$$

Note that the lower and upper bounds do not depend on s .

Take a another pair $T_*^+, T_*^- > 0$ such that

$$\sqrt{2T_*^-} \psi^- \left(\frac{x}{\sqrt{2T_*^-}} \right) \preceq u(x, \delta^2 T) \preceq \sqrt{2T_*^+} \psi^+ \left(\frac{x}{\sqrt{2T_*^+}} \right).$$

Lemma 4.2 implies that $T_*^+ > T_*^- \geq \delta T_*^+$. From time $\delta^2 T$, u will shrink to 0 in time $T_2 := (1 - \delta^2)T$, we consider another time interval: $t \in [\delta^2 T, \delta^2 T + \delta^2 T_2] = [\delta^2 T, T - (1 - \delta^2)^2 T]$, or $s \in [-\frac{1}{2} \log T - \frac{1}{2} \log(1 - \delta^2), -\frac{1}{2} \log T - \log(1 - \delta^2)]$. Replacing T^\pm by T_*^\pm in the above discussion we see that (4.13) holds on this time interval.

Repeat such processes infinite times we obtain the estimate (4.13) for w on $[0, T)$.

4.5 A priori estimate for w

The gradient bound of w is similar as that for ω in Section 2 and that for v in Section 3. Using the standard theory of parabolic equations (cf. [5, 6, 12, 13]) we can get the following conclusions.

Lemma 4.5 *Problem (4.11) with initial datum $w(\theta, -\frac{1}{2} \log T)$ (which is defined by (4.9) at $s = -\frac{1}{2} \log T$) has a unique, time-global solution $w(\theta, s) \in C^{2+\mu, 1+\mu/2}([-\theta_0, \theta_0] \times [-\frac{1}{2} \log T, \infty))$ and*

$$\|w(\theta, t)\|_{C^{2+\mu, 1+\mu/2}([-\theta_0, \theta_0] \times [-\frac{1}{2} \log T, \infty))} \leq C < \infty$$

where C depends only on μ, k_i, σ and β but not on t, T and u_0 .

The global existence of w is not new, it has been obtained from the existence of r on $[0, T)$ in Subsection 4.2. The estimate is important and will be used below.

4.6 Proof of Theorem 1.2

In this subsection we prove the existence of shrinking selfsimilar solutions on $[0, T)$. Conditions (1.10) and (1.13) imply that $\tilde{g}_1(s, w)$ and $\tilde{g}_2(s, w)$ are $\log b$ -periodic in s . A similar discussion as in Subsection 3.6 shows that there exists a $\log b$ -time-periodic solution \tilde{P} of (4.11) for $s \in \mathbb{R}$ with

$$(4.14) \quad \|\tilde{P}(\theta, s)\|_{C^{2+\mu, 1+\mu/2}([-\theta_0, \theta_0] \times \mathbb{R})} \leq C(\mu, k_1, k_2, \sigma, \beta)$$

such that $\|w(\cdot, s) - \tilde{P}(\cdot, s)\|_{C^2([-\theta_0, \theta_0])} \rightarrow 0$ as $s \rightarrow \infty$.

Now we recover $\tilde{P}(\theta, s)$ back to a corresponding solution of (1.1), that is, define $\tilde{U}, \tilde{\Xi}_i$ ($i = 1, 2$) by

$$(4.15) \quad e^{-s}e^{-\tilde{P}} \cos \theta = \tilde{U}(e^{-s}e^{-\tilde{P}} \sin \theta, T - e^{-2s}), \quad \tilde{\Xi}_i(T - e^{-2s}) = (-1)^i e^{-s}e^{-\tilde{P}(\theta_0, s)} \sin \theta_0$$

for $s \geq -\frac{1}{2} \log T$. They are well-defined as in previous subsections. Moreover,

$$\begin{aligned} b^{-1}\tilde{U}(e^{-s}e^{-\tilde{P}(\theta, s)} \sin \theta, T - e^{-2s}) &= b^{-1}e^{-s}e^{-\tilde{P}(\theta, s)} \cos \theta \\ &= e^{-s-\log b}e^{-\tilde{P}(\theta, s+\log b)} \cos \theta \\ &= \tilde{U}(e^{-s-\log b}e^{-\tilde{P}(\theta, s+\log b)} \sin \theta, T - e^{-2s-2\log b}) \\ &= \tilde{U}(b^{-1}e^{-s}e^{-\tilde{P}(\theta, s)} \sin \theta, T - b^{-2}e^{-2s}) \end{aligned}$$

and for $i = 1, 2$,

$$b^{-1}\tilde{\Xi}_i(T - e^{-2s}) = (-1)^i e^{-s-\log b}e^{-\tilde{P}(\theta_0, s+\log b)} \sin \theta_0 = \tilde{\Xi}_i(T - e^{-2s-2\log b}).$$

Hence, for $t' := e^{2s} \in (0, T]$ and $-\tilde{\Xi}_1(T - t') \leq x \leq \tilde{\Xi}_2(T - t')$ we have

$$b^{-1}\tilde{U}(x, T - t') = \tilde{U}(b^{-1}x, T - b^{-2}t'), \quad b^{-1}\tilde{\Xi}_i(T - t') = \tilde{\Xi}_i(T - b^{-2}t') \quad (i = 1, 2).$$

This means that $(\tilde{U}, \tilde{\Xi}_1, \tilde{\Xi}_2)$ is a shrinking selfsimilar solution of (1.1) on $[0, T)$. This proves Theorem 1.2. \square

4.7 Proof of Theorem 1.3

Now we assume $k_1 \equiv k_1(u)$, $k_2 \equiv k_2(u)$ and prove Theorem 1.3: the existence of shrinking selfsimilar solutions in $(-\infty, 0)$.

In the previous subsection we obtained a shrinking selfsimilar solution \tilde{U} of (1.1) on $[0, T)$ for any given $T > 0$. Moreover, for any $m < T$ and $t \in [T - m, T - \frac{1}{m}]$,

by the definition of new variables (4.8) we have $e^s \in [1/\sqrt{m}, \sqrt{m}]$. It follows from (4.15) and (4.14) that

$$(4.16) \quad \|\tilde{U}\|_{C^{2+\mu, 1+\mu/2}(\overline{Q}_{T-\frac{1}{m}} \setminus Q_{T-m})} \leq \tilde{C}, \quad \|\tilde{\Xi}_i\|_{C^{1+\mu/2}([T-m, T-\frac{1}{m}])} \leq \tilde{C},$$

for some $\tilde{C} = \tilde{C}(\mu, k_1, k_2, \sigma, \beta, m)$ independent of t and T , where Q_{t_0} , for any $t_0 > 0$, is defined in the beginning of Section 2.

Denote $\hat{t} := t - T$ and

$$W(x, \hat{t}; T) := \tilde{U}(x, T + \hat{t}), \quad \Upsilon_i(\hat{t}; T) := \tilde{\Xi}_i(T + \hat{t}) \quad (i = 1, 2), \quad \text{for } -T \leq \hat{t} < 0.$$

Then

$$(4.17) \quad \begin{cases} W_{\hat{t}}(x, \hat{t}) = a(W_x(x, \hat{t}))W_{xx}(x, \hat{t}), & -\Upsilon_1(\hat{t}; T) < x < \Upsilon_2(\hat{t}; T), \quad -T \leq \hat{t} < 0, \\ W_x(x, \hat{t}) = -k_1(W(x, \hat{t})), \quad W(x, \hat{t}) = -x \tan \beta & \text{for } x = -\Upsilon_1(\hat{t}; T), \quad -T \leq \hat{t} < 0, \\ W_x(x, \hat{t}) = k_2(W(x, \hat{t})), \quad W(x, \hat{t}) = x \tan \beta & \text{for } x = \Upsilon_2(\hat{t}; T), \quad -T \leq \hat{t} < 0. \end{cases}$$

For any fixed $m \in \mathbb{N}$, consider $W(x, \hat{t}; T)$ for $T > m$, then (4.16) implies that

$$\|W(x, \hat{t}; T)\|_{C^{2+\mu, 1+\mu/2}(\overline{Q}_{-m} \setminus \mathcal{Q}_{-\frac{1}{m}})} \leq \tilde{C}, \quad \|\Upsilon_i(\hat{t}; T)\|_{C^{1+\mu/2}([-m, -\frac{1}{m}])} \leq \tilde{C},$$

where $i = 1, 2$ and for any $l > 0$,

$$\mathcal{Q}_{-l} := \{(x, \hat{t}) \mid -\Upsilon_1(\hat{t}; T) < x < \Upsilon_2(\hat{t}; T), \hat{t} \in (-l, 0)\}.$$

So there exist functions $\hat{U}(x, \hat{t}) \in C^{2+\mu, 1+\mu/2}(\overline{Q}_{-m} \setminus \mathcal{Q}_{-\frac{1}{m}})$, $\hat{\Xi}_1, \hat{\Xi}_2 \in C^{1+\mu/2}([-m, -\frac{1}{m}])$ and a subsequence $\{T_j\}$ of $\{T\}$ such that, as $j \rightarrow \infty$,

$$\|W(x; \hat{t}; T_j) - \hat{U}(x, \hat{t})\|_{C^{2,1}(\overline{Q}_{-m} \setminus \mathcal{Q}_{-\frac{1}{m}})} \rightarrow 0, \quad \|\Upsilon_i(\hat{t}; T_j) - \hat{\Xi}_i(\hat{t})\|_{C^1([-m, -\frac{1}{m}])} \rightarrow 0.$$

Using Cantor's diagonal argument, there exist functions $\hat{U}(x, \hat{t}) \in C^{2+\mu, 1+\mu/2}(\mathcal{Q}_{-\infty})$, $\hat{\Xi}_i \in C^{1+\mu/2}(-\infty, 0)$ ($i = 1, 2$) and a subsequence of $\{T\}$ (denoted it again by $\{T_j\}$) such that, as $j \rightarrow \infty$,

$$(4.18) \quad \|W(x; \hat{t}; T_j) - \hat{U}(x, \hat{t})\|_{C_{\text{loc}}^{2,1}(\mathcal{Q}_{-\infty})} \rightarrow 0, \quad \|\Upsilon_i(\hat{t}; T_j) - \hat{\Xi}_i(\hat{t})\|_{C_{\text{loc}}^1(-\infty, 0)} \rightarrow 0,$$

where $\mathcal{Q}_{-\infty} := \{(x, \hat{t}) \mid -\hat{\Xi}_1(\hat{t}) < x < \hat{\Xi}_2(\hat{t}), \hat{t} \in (-\infty, 0)\}$. Hence $(W, \Upsilon_1, \Upsilon_2) = (\hat{U}, \hat{\Xi}_1, \hat{\Xi}_2)$ is a solution of (4.17) for all $\hat{t} \in (-\infty, 0)$, that is, a classical solution of (1.17).

Finally, we prove the similarity of $(\hat{U}, \hat{\Xi}_1, \hat{\Xi}_2)$. By the similarity of $(\tilde{U}, \tilde{\Xi}_1, \tilde{\Xi}_2)$ in the previous subsection and by the definitions of W and Υ_i it is easily seen that

$$W(x, \hat{t}; T) = bW(b^{-1}x, b^{-2}\hat{t}; T), \quad \Upsilon_i(\hat{t}; T) = b\Upsilon_i(b^{-2}\hat{t}; T) \quad (i = 1, 2)$$

for $-T \leq \hat{t} < 0$. Therefore, (4.18) implies that \hat{U} and $\hat{\Xi}_i$ satisfy conditions (1.15) and (1.16) in $(-\infty, 0)$, respectively. This prove Theorem 1.3. \square

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